Temperature Distribution in a Fin Partially Cooled by Nucleate Boiling

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When heat is transferred from a fin into a nearly saturated liquid, nucleate boiling may set in if the temperature on the fin is sufficiently higher than the liquid's saturation temperature (*1*). Such a mode of heat transfer would be very efficient because of the high heat transfer coefficient of nucleate boiling over an extended surface area. Prior to this report, only complicated numerical solutions for nucleate boiling from a fin had been proposed (*1, 2*). The purposes of this report are to present a simplified model for establishing a criterion for the incipience of nucleate boiling on a straight fin, to propose an expression for the length of the boiling section, and to determine the heat flux for a given base temperature. The simplified method will provide a quick estimate without the need of resorting to a tedious step-by-step numerical calculation with difference equations.

The temperature profile on a straight fin was determined, and this temperature was compared with the existing criterion for the incipience of nucleate boiling. The temperature for the incipience of nucleate boiling can be determined either experimentally or by use of the equation proposed by Hsu (*3*):

$$\theta_i = \theta_{sat} + \frac{2AC_3}{\delta} + \sqrt{\left(2\theta_{sat} + \frac{2AC_3}{\delta}\right)\left(2\frac{AC_3}{\delta}\right)} \quad (1)$$

In the present study, the temperature profile distribution along the straight fin was derived to determine the location where the criterion for the incipience of nucleate boiling is satisfied. Once the length of the boiling section is known, the heat flux at the base can be determined.

A simple experiment was conducted to check the theory. Also, a comparison was made between the present theory and the experimental data on nucleate boiling from a fin reported by Haley and Westwater (*1*).

ANALYSIS

If nucleate boiling does occur on a fin, it will first take place at the root of the fin. As the temperature of the fin is increased, the nucleate boiling section will extend out further and further on the fin. The one fin can thus be considered as made up of two separate fins in contact with each other. The section near the root is hot enough to have nucleate boiling taking place while the end section is being cooled by convection (which could be either forced or free convection). The model is shown in Figure 1 for a straight fin of rectangular shape. Similar analyses can be made on other geometries, such as a cylindrical fin, without much additional difficulty.

The fin could be either vertically or horizontally oriented and extended into either a cross flow or a pool. This variation of geometry should be properly considered only when the value of the average heat transfer coefficient is selected. The conventional assumptions used for the derivation of the temperature profile on a straight fin were applied herein, namely, constant properties, flat profile in y direction, and negligible end effect. The differential equation for a straight fin is

$$\frac{d^2\theta}{dx^2} = \frac{h(\theta)}{Kb}\theta \quad (2)$$

Consider the convective section first, and assume that the heat transfer coefficient is represented by a constant value h_c.

$$\frac{d^2\theta}{dx^2} = \frac{h_c}{Kb}\theta \quad (2a)$$

The boundary conditions are

$$\theta = \theta_i \text{ at } x = L_i$$

$$\frac{d\theta}{dx} = 0 \text{ at } x = L$$

Use of Equation (2*a*) and the previous boundary conditions gives the classical fin equation for a fin of height $L - L_i$.

$$\theta = \theta_i \frac{\cosh (L-x)\sqrt{\frac{h_c}{Kb}}}{\cosh (L-L_i)\sqrt{\frac{h_c}{Kb}}} \quad \text{for} \quad L_i \leq x \leq L \quad (3)$$

The heat flux at $x = L_i$ is

$$-K\left(\frac{d\theta}{dx}\right)_i = q_i$$

$$= \sqrt{\frac{h_c K}{b}}\,\theta_i \tanh\left[\sqrt{\frac{h_c}{Kb}}(L-L_i)\right] \quad (4)$$

For the boiling section, the heat transfer coefficient varies as the temperature difference θ to the n^{th} power. Thus, Equation (2) should take the form

$$\frac{d^2\theta}{dx^2} = \frac{h_i\left(\frac{\theta}{\theta_i}\right)^n}{Kb}\theta \quad (2b)$$

where h_i and θ_i are the heat transfer coefficient and the temperature, respectively, at the intersection of the two sections; n is usually in the range of 2 to 4. Multiplying both sides of Equation (2*b*) by $d\theta/dx$ yields

$$\frac{d^2\theta}{dx^2}\frac{d\theta}{dx} = \frac{h_i}{Kb\theta_i^n}\theta^{n+1}\frac{d\theta}{dx} \quad (5)$$

or

$$\frac{d}{dx}\left[\left(\frac{d\theta}{dx}\right)^2\right] = \frac{2h_i}{Kb\theta_i^n(n+2)}\frac{d(\theta^{n+2})}{dx} \quad (5a)$$

Integration of Equation (5*a*) from the intersection point where $\theta = \theta_i$ and $d\theta/dx = (d\theta/dx)_i$ to the arbitrary point where $\theta = \theta$ and $d\theta/dx = d\theta/dx$ produces the following expression for $d\theta/dx$:

$$\frac{d\theta}{dx} = -\sqrt{\frac{2}{n+2}m_i^2\theta_i^2\left[\left(\frac{\theta}{\theta_i}\right)^{n+2} - 1\right] + \left(\frac{d\theta}{dx}\right)_i^2} \quad (6)$$

where

$$m_i = \sqrt{\frac{h_i}{Kb}} \quad (7)$$

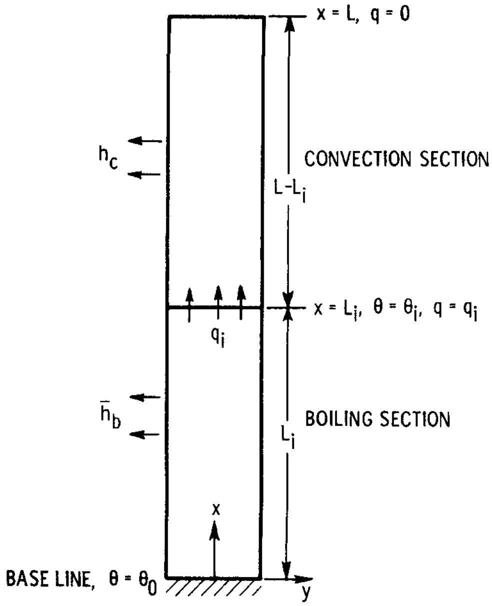

Fig. 1. Simplified model for coexistence of nucleate boiling and convection on a fin.

The temperature profile cannot be expressed in an explicit form. However, the length of the boiling section, $\int_0^{L_i} dx$, can be expressed as

$$L_i = \int_0^{L_i} dx = \int_{\theta_i}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2}{2+n} m_i^2\ \theta_i^2 \left[\left(\frac{\theta}{\theta_i} \right)^{n+2} - 1 \right] + \left(\frac{d\theta}{dx} \right)_i^2}} \quad (8)$$

This equation generally cannot be evaluated in terms of simple well-known functions. For example, the case $n = 2$ in the integral equation [Equation (8)] would yield an elliptic integral and thus its solution would involve elliptic functions; for $n = 3, 4$, etc., the integral equation would yield hyperelliptic integrals and thus would involve hyperelliptic functions as solutions. However, for an arbitrary value of n (for example, between 2 and 4), only numerical or graphical integration can be used on Equation (8). Furthermore, the term $(d\theta/dx)_i$ has to be known first, which, as shown in Equation (4), is a function of $L - L_i$. Therefore Equation (8) is really not an explicit form of L_i. It can be solved only by iteration. Some approximate solutions to Equations (6) and (8) can be obtained, though, when θ_0 and/or n are so large that

$$\frac{2}{n+2} m_i^2\ \theta_i^2 \left[\left(\frac{\theta_0}{\theta_i} \right)^{n+2} - 1 \right]$$

$$\approx \frac{2}{n+2} \left(\frac{\theta_0}{\theta_i} \right)^{n+2} m_i^2\ \theta_i^2 >> \left(\frac{\partial \theta}{\partial x} \right)_i^2 \quad (9)$$

or, from Equation (4)

$$\frac{2}{n+2} \left(\frac{\theta_0}{\theta_i} \right)^{n+2} >> \gamma_i^2 \tanh^2 [\gamma_i\, m_i\, (L - L_i)] \quad (9a)$$

where

$$\gamma_i = \sqrt{\frac{h_c}{h_i}}$$

Then

$$\left(\frac{d\theta}{dx} \right)_0 \approx - \sqrt{\frac{2}{n+2} m_i^2\ \theta_i^2 \left[\left(\frac{\theta_0}{\theta_i} \right)^{n+2} \right]} \quad (6a)$$

and

$$L_i \approx \int_{\theta_i}^{\theta_0} \frac{d\theta}{m_i \theta_i \sqrt{\frac{2}{n+2}} \left(\frac{\theta}{\theta_i} \right)^{(n+2)/2}}$$

$$= \frac{1}{m_i} \sqrt{\frac{2+n}{2}} \int_1^{(\theta_0/\theta_i)} \left(\frac{\theta}{\theta_i} \right)^{-(n+2)/2} d\left(\frac{\theta}{\theta_i} \right)$$

$$= \frac{2}{m_i n} \sqrt{\frac{2+n}{2}} \left[1 - \left(\frac{\theta_i}{\theta_0} \right)^{n/2} \right] \quad (8a)$$

Since the object of this paper is to provide some simple method of estimating the heat transfer performance of a fin, the involvement of iteration and numerical integration defeats the objective. Some further approximation is then needed. One simple approach is to assume that a mean

TABLE 1.

Liquid	Boundary between convection and nucleate boiling regions		Exponent in nucleate boiling region	Boundary between nucleate boiling and transition boiling regions		Exponent in transition boiling region
	θ_i, °F.	m_i^2, ft.$^{-2}$	n	θ_p, °F.	m_p^2, ft.$^{-2}$	n_t
Isopropanol	25	262	5	35	2620	—1.75
Freon	15	52.4	3.1	45	1746	—4.0

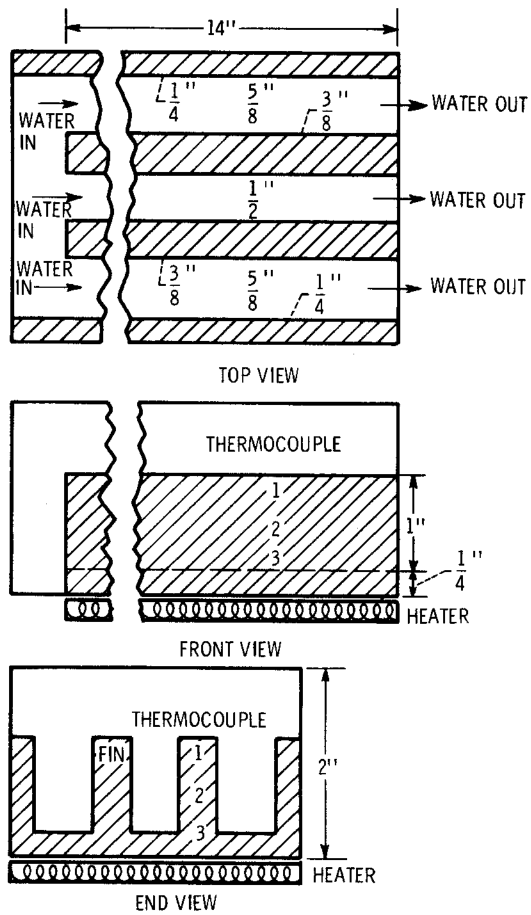

Fig. 2. Geometry of test section (body is made of single block of aluminum).

heat transfer coefficient $\overline{h}_b$ can be used in Equation (2*b*) in place of $h_i(\theta/\theta_i)^n$:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\overline{h}_b}{Kb}\theta \tag{2c}$$

The boundary conditions are

$$x = 0, \quad \theta = \theta_0$$

$$x = L_i, \; q_i = \sqrt{\frac{h_c K}{b}}\,\theta_i \tanh\left[(L - L_i)\sqrt{\frac{h_c}{Kb}}\right]$$

$$= \gamma_i m_i K\theta_i \tanh\left[\gamma_i m_i (L - L_i)\right]$$

The second boundary condition is needed to match the heat flux at the upper end of the boiling section with that at the lower end of the convective section as given by Equation (4). The solution to Equation (2*c*), by using the previous two boundary conditions, gives the temperature profile for the section exposed to boiling:

$$\theta = \theta_0 e^{mx} + M(e^{-mx} - e^{mx}) \tag{10}$$

where

$$M = \frac{m\theta_0 e^{mL_i} + \sqrt{\frac{h_c}{Kb}}\,\theta_i \tanh\left[(L - L_i)\sqrt{\frac{h_c}{Kb}}\right]}{m(e^{mL_i} + e^{-mL_i})} \tag{11}$$

$$m = \sqrt{\frac{\overline{h}_b}{Kb}} \tag{12}$$

The temperature at the point where the boiling ceases is

$$\theta_i = \frac{\theta_0 e^{mL_i}\left[1 - \tanh(mL_i)\right]}{1 + \sqrt{\frac{h_c}{h_b}}\tanh\left[\sqrt{\frac{h_c}{\overline{h}_b}}\,m(L - L_i)\right]\tanh(mL_i)} \tag{13}$$

Or, in the form of a dimensionless equation

$$\frac{\theta_i}{\theta_0} = \frac{1}{\cosh \beta\mu}\left(\frac{1}{1 + \gamma\tanh\left[\gamma\mu(1-\beta)\right]\tanh\mu\beta}\right) \tag{14}$$

where

$$\beta = \frac{L_i}{L}$$

$$\mu = mL$$

$$\gamma = \sqrt{\frac{h_c}{\overline{h}_b}}$$

Thus, if the temperature required for the incipience of boiling θ_i is known, either experimentally or from Equation (1), Equation (14) can be used to determine the dimensionless length β of the boiling section for a specified θ_0, μ, and γ.

The heat flux at the base of a fin may be determined if the temperature gradient at the base is known. The temperature gradient can be obtained by differentiation of Equation (10) when $x = 0$. With the newly defined notations

$$\left.\frac{d\theta}{dx}\right|_{x=0} = -m\theta_0 \tanh mL_i - \frac{\theta_i m\gamma \tanh\left[(L - L_i)\,m\gamma\right]}{\cosh(mL_i)} \tag{15}$$

Equation (15), in conjunction with Equation (14), shows that the temperature gradient is a function of h_c, $\overline{h}_b$, K, b, L, θ_0, and L_i. All these entries are given conditions except L_i, which has to be determined from Equation (14).

Although the analysis is derived for the coexistence of the convective and nucleate boiling modes on the surface of a straight fin, it could be readily extended to other cases where two or more different modes of heat transfer coexist. Thus, the analysis is equally applicable to such cases as the coexistence of film transition or transition nucleate boiling on a fin.

EXPERIMENTAL APPARATUS AND PROCEDURE

A simple experimental setup was constructed to test the theory. Because of the crudeness of the instrumentation and the control system, only a qualitative comparison was attainable.

The test section consisted of two parallel aluminum fins inserted in a straight channel as shown in Figure 2. The straight fins are 14 in. long, ⅜ in. thick, and 1 in. high. Near the downstream end of one of the fins, three chromel-alumel thermocouples (AWG 24) were inserted spaced ½ in. apart, one each at the root, center, and top of the fin, respectively. Another thermocouple was inserted into the main stream to record the bulk temperature of the flow. The aluminum bottom of the channel is ¼ in. thick and is heated by an electric heating plate rated at 1,200 w. Good insulation was provided to reduce the heat loss. Water heated to near its boiling point was supplied from a preheater. The flow rate

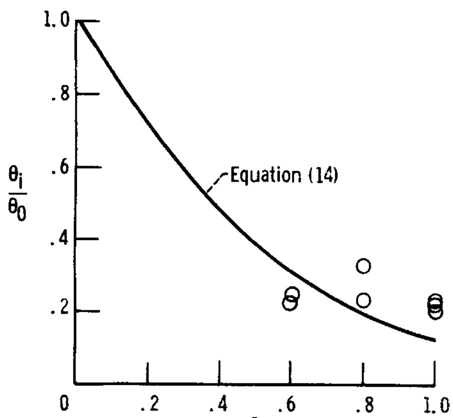

Fig. 3. Comparison of theoretical and experimental results for water boiling on aluminum rectangular fin. μ, 2.72; γ, 0.50; h_b is assumed constant.

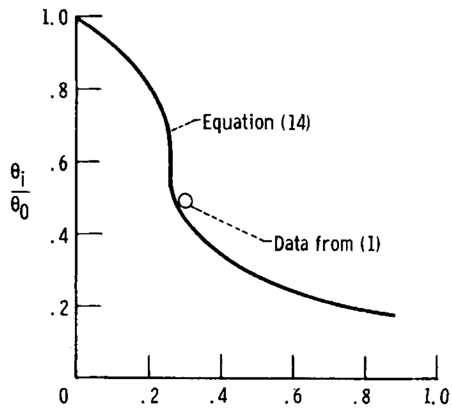

Fig. 4. Comparison of theoretical and experimental results for isopropanol boiling on a cylindrical fin. L, 1.207 in.; b = 4/2 = 1/16 in.; K, 220 B.t.u./(hr.)(ft.)(°F.); h_b varies with θ_0.

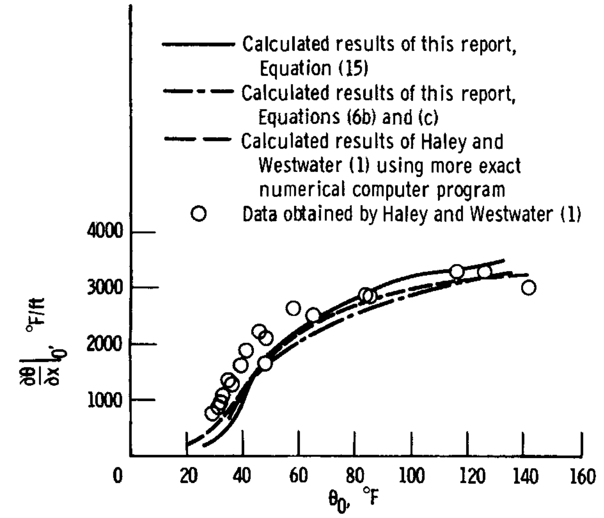

Fig. 5. Temperature gradient at base of cylindrical fin for boiling of isopropanol. *L*, 1.207 in.; *r*, 0.125 in.

was measured by collecting water at the exit of the channel. A potentiometer and a pyrometer were used to take thermocouple readings.

Some preliminary runs were made to determine the surface temperature required for the incipience of boiling under similar flow conditions. Roughly speaking, the average wall temperature required for incipience of boiling was 241°F. for an average bulk temperature of 202°F. When Equation (1) was used, it was found that the corresponding value of δ is 5×10^{-5} ft. The corresponding temperature for incipience of boiling when the bulk temperature is at saturation temperature is calculated to be 236°F.

When a steady state condition was achieved with nucleate boiling occurring over part of the fin, the height of the boiling section was measured with a ruler, and the temperatures were recorded.

RESULTS AND DISCUSSION

Data from This Study

The experimental results are shown in Figure 3. Corresponding theoretical values of the length of the boiling section [from Equation (14)] are also shown for comparison. The following values of the parameters were used in Equation (14).

h_c = 500 B.t.u./(hr.)(sq.ft.)(°F.)
$\overline{h}_b$ = 2,000 B.t.u./(hr.)(sq.ft.)(°F.) (Note: The value of h_b should really be a function of θ_0, but for small range of θ_0, it is assumed to be constant.)
K = 120 B.t.u./(hr.)(ft.)(°F.)
L = 1/12 ft.
b = 1/64 ft.
μ = 2.72
γ = 0.50
θ_i = 24°F.

The h_c and $\overline{h}_b$ values were those deducible from the low range of the typical boiling curves reported in the literature.

Since the facilities were limited and the analytical model was highly simplified, comparison can only be expected on the basis of a first-order approximation. In view of such limitations, the agreement between the experimental and the theoretical values is reasonably good. More refined experiments should be carried out to verify the result further.

Comparison with Results of Haley and Westwater

An interesting comparison can also be made with the experimental results of Haley and Westwater (*1*). Figure 1 of reference *5* showed that nucleate boiling of isopropanol covered about one-third of the copper fin. The reported data were L = 1.207 in., θ_0 = 52°F., and r = 0.125 in. Equation (14) may be applied to a cylindrical fin when b is replaced by $r/2$. The experimental point is shown for comparison in Figure 4. The value of θ_i = 25°F. reported in reference *4* was used. General agreement between experiment and theory is observed.

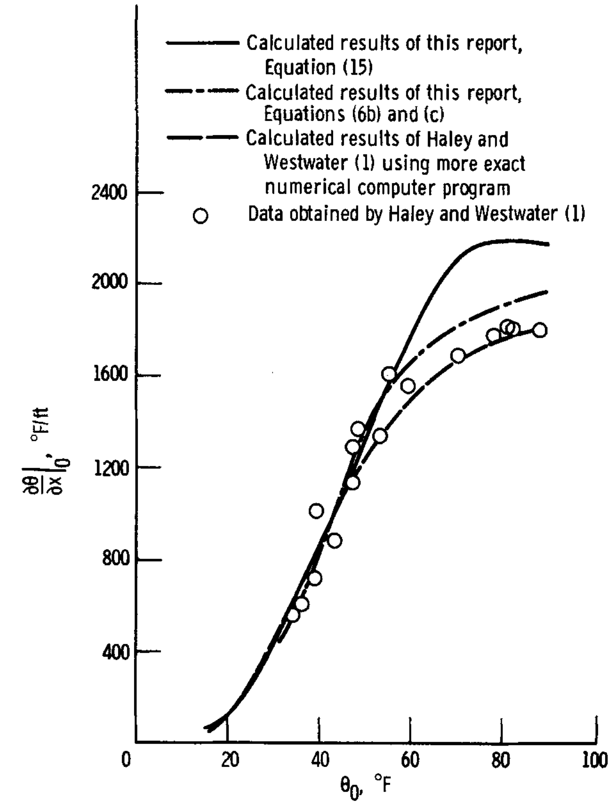

Fig. 6. Temperature gradient at base of cylindrical fin for boiling of freon. *L*, 1.207; *r*, 0.125.

Haley and Westwater also reported experimental data on the temperature gradient at the base of a fin $(\partial\theta/\partial x)_0$ for the boiling of isopropanol and of freon. These data are compared with the values calculated from Equation (15) as shown in Figures 5 and 6. Also shown in these figures are the analytical values computed by Haley and Westwater using step-by-step numerical computation. For isopropanol, Equation (15) underpredicted the experimental values but gave a correct trend. The result for freon is more encouraging. The experimental data fell on the theoretical curves in the lower range of the base temperature but began to depart at the higher end. Apparently, the departure is due to the drastic change of h with θ near the upper end of the nucleate boiling region which cannot be represented properly by an average h.

For comparative purposes, the temperature gradients at the fin base are also computed with the n^{th}-power model [Equations (2*b*) and (6)]. The need of the n^{th}-power model in lieu of the simpler mean heat transfer coefficient model [Equation (2*c*)] exists only in the region of large θ. Since the criterion, Equation (9), is also valid in the large θ range, Equation (6*a*) can be used as an approximation to Equation (6) with an error no greater than 2% for the present cases of boiling of isopropanol and freon. If the value of θ_0 is too high, transition boiling sets in at the root section of the fin. Therefore the model has to be revised to consider both the nucleate boiling and the transition boiling sections.

$$\left(\frac{\partial\theta}{\partial x}\right)_0 = -\sqrt{\frac{2}{n_t+2} m_p^2\ \theta_p^2 \left[\left(\frac{\theta_0}{\theta_p}\right)^{n_t+2} - 1\right] + \left(\frac{\partial\theta}{\partial x}\right)_p^2} \qquad (6b)$$

$$\left(\frac{\partial\theta}{\partial x}\right)_p \cong -\sqrt{\frac{2}{n+2} m_i^2\ \theta_i^2 \left[\left(\frac{\theta_p}{\theta_i}\right)^{n+2} - 1\right]} \qquad (6c)$$

Equation ($6b$) is similar to Equation (6) with the exception that n, θ_i, and m_i are replaced by n_t, θ_p and m_p, respectively. Equation ($6c$) is an approximation similar to Equation ($6a$) wherein θ_p and $(\partial\theta/\partial x)_p$ replace θ_0 and $(\partial\theta/\partial x)_0$.

The parameter values for isopropanol and freon are given in Table 1. The calculated results from Equations ($6b$) and ($6c$) are shown in Figures 5 and 6. Agreement with experimental data is greatly improved.

Thus the simplified method proposed in this report can be used to estimate the temperature gradient at the fin base to the first-order approximation, or it can be used to provide the initial values for a more elaborate computer program such as that used in reference *1*. If moderate accuracy is adequate, then Equation ($6a$) may be used, which still is vastly more simple than the exact numerical solution.

CONCLUSION

A simple model is proposed to determine the length of the nucleate boiling section on a fin and the base heat flux of the fin as a function of the ratios of heat transfer coefficients of nucleate boiling and convective modes, as well as the characteristics of the fin. Qualitative agreement is observed between the analytical and the experimental results.

ACKNOWLEDGMENT

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NOTATION

A = $2\sigma T_{sat}/\lambda\rho_v$
b = half-width of fin
C = coefficient
C_3 = constant, 1.6
h = heat transfer coefficient
K = thermal conductivity
L = length of fin
L_i = length of boiling section
M = parameter, Equation (11)
m = $\sqrt{\bar{h}_b/Kb}$
m_i = $\sqrt{\bar{h}_i/Kb}$
n = exponent used in Equation ($2b$)
q = heat flux
r = radius
T = temperature
T_{bulk} = bulk temperature
x = coordinate shown in Figure 1
y = coordinate shown in Figure 1

Greek Letters

β = L_i/L
γ = $\sqrt{\bar{h}_c/\bar{h}_b}$
δ = limiting thermal layer thickness
θ = $T - T_{bulk}$
λ = latent heat of evaporation
μ = mL
ρ_v = density of saturation vapor
σ = surface tension

Subscripts

b = boiling
c = convection
i = incipience of boiling or transition between convection and nucleate boiling
0 = base
p = transition from nucleate to transition boiling
sat = saturation
t = transition boiling region

Superscript

— = average value

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Free Oscillations of Fluids in Manometers

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Biery (*1, 2*) recently presented the results of a numerical and experimental study of free oscillations of fluids in U shaped manometer tubes. His work included the use of Newtonian and non-Newtonian fluids. In order to obtain good agreement between calculations and experiments, Biery found it necessary to introduce a correction for end effects which was based upon physical concepts but which could not be evaluated on its own; the heuristic correction